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Recirculation within a fluid sphere at moderate Reynolds numbers

By D. A. BARRY AND J.-Y. PARLANGE†

School of Civil and Environmental Engineering and Contaminated Land Assessment and Remediation Research Centre, The University of Edinburgh, Edinburgh EH9 3JN, UK

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Motion of a single fluid sphere is described by two theories, each characterized by different levels of Hill's vortex circulation within the sphere. An existing experimental data set giving measurements of vertical velocity along the major axis of the sphere is re-examined. Contrary to published discussions of that experiment, we find that the theory of Parlange agrees better with the laboratory data than that of Harper & Moore. This agreement supports the key difference between the two theories, i.e. that the fluid within the sphere is unlikely to have a singular (infinite) velocity as it moves upwards towards the stagnation region at the top of the sphere.

1. Introduction

Steady-state motion of a fluid sphere in a quiescent medium induces recirculation within the sphere. Normally, hydrodynamic forces and surface tension will affect the shape of the interface, but we assume that shape is near spherical, e.g. because of surface tension effects. We also assume that the Reynolds number is of a magnitude such that viscous effects are important only in thin boundary layers and that the recirculating flow, outside the boundary layers, is that of a Hill's vortex (Harper & Moore 1968; Parlange 1970). Both Harper & Moore (1968) and Parlange (1970) give an estimate of the strength of the vortex. The former authors assume that the fluid within the sphere moving upwards from the rear stagnation region retains its vorticity all the way to the front stagnation region, so that the velocity eventually has a singularity and becomes infinite. On the other hand, Parlange (1970) presented an alternative model based on the assumption that this singularity is not physically feasible and that vorticity possibly diffuses away in spite of low viscous effects, because the fluid spends a relatively long time in the front stagnation region of the bubble. One can speculate that other physical processes might also be responsible for erasing the singularity.

At first sight, at least, it would seem obvious to compare the models of Harper & Moore (1968) and Parlange (1970) from terminal velocity measurements of rising bubbles. Unfortunately, it is not possible to distinguish the accuracy of the models from such measurements as the predicted drags are very similar. Yet the predicted circulations are significantly different, typically by a factor slightly larger than two, for example. Consequently, Parlange (1970) suggested that measurement of the recirculation would be an excellent means to evaluate the two models. These measurements were obtained by Bhaga & Weber (1981) who carried out careful experiments in

† Permanent address: Department of Biological and Environmental Engineering, Cornell University, Ithaca, NY 14853-5701, USA.

the wake of spherical cap bubbles. Interestingly, the wake is near spherical at large enough Reynolds numbers but still in the laminar regime, as first discussed by Parlange (1969) (cf. Wegener & Parlange 1973). Bhaga & Weber (1981) stated, as illustrated in their figure 20, that 'the data are well fitted by the Harper & Moore theory'. Consequently, they concluded that 'the assumption of Harper & Moore (1968) that the fluid... retains its vorticity... is more reasonable than the vorticity destruction assumption of Parlange (1970)'. Based on those crucial experiments, the theory of Harper & Moore (1968) is currently preferred to that of Parlange (1970) (e.g. Dandy & Leal 1989; Fan & Tsuchiya 1990; Hoffmann & van den Bogaard 1995; Hoogstraten *et al.* 1998).

Since Bhaga & Weber (1981) carried out exactly the experiments for which the two theories make significantly different predictions, the results seemed conclusive. However, recently we had occasion to re-examine their experimental results carefully. In particular, we consider predictions of recirculation within the wakes without (§ 2) and with (§ 3) boundary layer corrections. Two points concerning their theoretical analysis became apparent:

1. Bhaga & Weber (1981) used the wrong Reynolds number in calculating the new Hill's vortex versions of the theories of Harper & Moore (1968) and Parlange (1970); and, much more importantly.
2. They did not apply boundary layer corrections, as required by both theories.

The second issue is explored in detail below.

2. Spherical cap experiments

We generally follow the notation of Parlange (1970). The recirculating flow within a spherical drop or bubble is a Hill's vortex, with stream function, Ψ_i. The theory of Parlange (1970) gives, for Ψ_i,

$$\Psi_i = \frac{3ur^2\sin^2\theta(r^2-a^2)}{4a^2}\left[1-\left(1+\frac{3}{2}\frac{\mu_i}{\mu_o}\right)\frac{4}{(3\pi R_o)^{1/2}}\right]. \tag{2.1}$$

Here, subscript i refers to values inside the drop and o outside, μ is the viscosity, u the (steady) velocity of the drop, R_o the Reynolds number ($2\rho ua/\mu$), ρ the density, a the radius of the drop, r and θ the usual spherical coordinates (see figure 1). Note that, for the spherical cap experimental data considered below, the material properties inside and outside the wake are identical.

For the wake of a spherical cap bubble then, (2.1) reduces to

$$\Psi_i = \frac{3ur^2\sin^2\theta(r^2-a^2)}{4a^2}\left[1-\frac{10}{(3\pi R_o)^{1/2}}\right]. \tag{2.2}$$

The corresponding expression of Harper & Moore (1968) is

$$\Psi_{iHM} = \frac{3ur^2\sin^2\theta(r^2-a^2)}{4a^2}\left[1+C\left(\frac{2}{R_o}\right)^{1/2}\right]. \tag{2.3}$$

The coefficient, C, of the $(2/R_o)^{1/2}$ correction in (2.3) is obtained from the solution of the integral equation (3.23) of Harper & Moore (1968). We solved their integral equation numerically and found that the difference between the empirical approximation they suggest for $2^{1/2}C$ and its exact value is insignificant, i.e. -7.5 versus -7.46762. Although there is very little difference between these values, we use the exact value in the following.

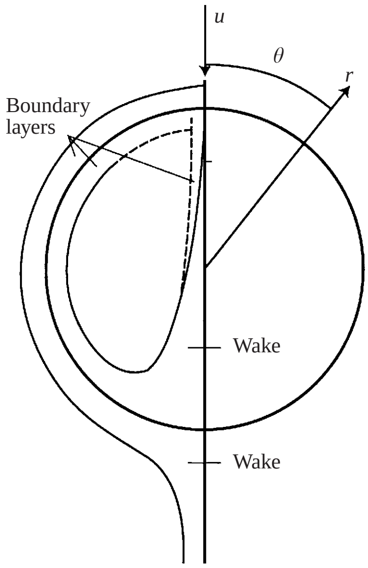

FIGURE 1. Schematic outline of a fluid sphere (radius a), showing the coordinate system used and boundary layers, the latter indicated by dashes in the front stagnation region for the internal boundary layer, where they disappear. The inner and outer boundary layers produce the inner and outer wakes when they reach the rear of the sphere ($\theta = \pi$).

The correction of order $R_o^{-1/2}$ in (2.2) is less than half the correction in (2.3) so that, as noted above, the two theories are significantly different in this respect. In the experiment of Bhaga & Weber (1981), the Reynolds number based on the gas bubble equivalent diameter is 94. R_o, however, requires knowledge of the radius of the wake a. There is some ambiguity in estimating a as the wake behind the spherical cap is not exactly a sphere as is clear, for example, from figure 19(d) of Bhaga & Weber (1981). For instance, on that figure, the position of the closed wake suggests $a \simeq 2.61$ cm or $R_o \simeq 188$.

Bhaga & Weber (1981) measured the vertical velocity along the line $\theta = \pi/2$ (data shown in figure 2). Note that, in line with the flow direction shown in figure 1, the sign of the normalized velocity, q/u, is opposite to that used by Bhaga & Weber (1981). Equations (2.2) and (2.3) can each be used to predict a velocity, ignoring boundary layer corrections, from

$$q_{i\theta} = \frac{1}{r}\frac{\partial \Psi}{\partial r}. \tag{2.4}$$

Both predictions are shown in figure 2 for $R_o = 188$. We show also in that figure the uncorrected Hill's vortex, i.e. for $r/a \leqslant 1$, either (2.2) or (2.3) without the $O(R_o^{-1/2})$ corrections or, for $r/a \geqslant 1$, (3.5) below, again without the correction term. In the various figures presented subsequently, the difference between the classical Hill's vortex prediction and various theoretical predictions is negligible for $r/a \geqslant 2$, and so all plots are truncated at $r/a = 2$.

From figure 2, we observe that in the region of positive velocities there is little to justify preferring one result over the other: one is above and one below the data by about the same amount. It is only near the axis that Harper & Moore (1968) appears to fit the data far better. Note also that as the flow is axisymmetric the volume of fluid in that region is relatively small.

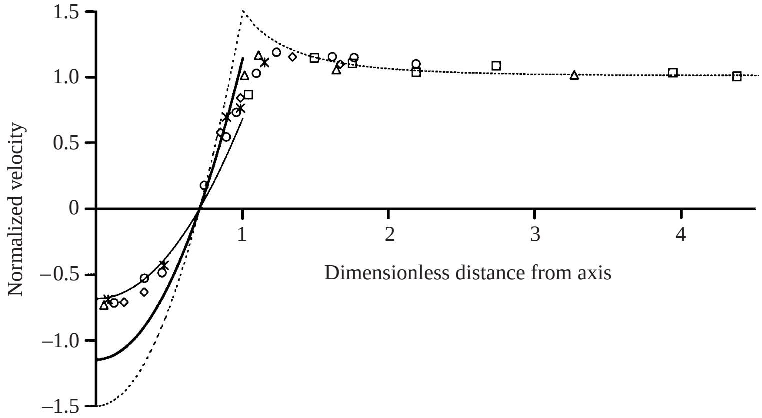

FIGURE 2. Vertical velocity measurements (symbols) of Bhaga & Weber (1981); together with the Hill's vortex: dashes; predictions of Parlange (1970): thick line, (2.2); and predictions of Harper & Moore (1968): thin line, (2.3). All predictions based on $R_o = 188$.

As can be seen from figure 2, (2.2) predicts a relative velocity of -1.14 for $r/a = 0$, which is below the measured value of around -0.73. By comparison, (2.3) predicts the slightly higher velocity of -0.68 which is in fact consistent with the numerical results of Hoffmann & van den Bogaard (1995) (see their figure 7). However, Bhaga & Weber (1981), in their application of (2.2), suggest a velocity of around -1.05 at $r/a = 0$, i.e. they take R_o to be as low as 120, which is clearly incorrect, as we remarked in § 1. Taking this value slightly improves the comparison of (2.2) with the experimental data (see their figure 20). Figure 2, by contrast, shows the appropriate prediction of (2.2) computed taking $R_o = 188$. Their error, however, does not affect the following analysis at all and has no impact on the discussion. On the other hand, in their application of (2.3), Bhaga & Weber (1981) use the more realistic value of R_o of around 212. This value (i.e. 212) is also consistent with the numerical results of Hoffmann & van den Bogaard (1995) (see their table 1) suggesting that we should have $R_o = 94 \times 2.26 \simeq 212$. It is unclear what value to take for R_o. This is in part due to the fact that the recirculating flow is not exactly spherical and it is likely that we could reasonably take R_o as 188 or 212. In the following we systematically consider both values of R_o and so check on the sensitivity of the results to this uncertainty.

So far only the Hill's vortex solutions have been discussed. It is interesting that in the application of the theoretical results neither Bhaga & Weber (1981) nor Hoffmann & van den Bogaard (1995) attempted to consider the effect of boundary layer corrections on the theoretical predictions. Such corrections are crucial to the analysis of the experimental data, the details of which follow.

3. Boundary layer corrections

Taking into account boundary layers for $r/a \simeq 1$, the theory of Parlange (1970) gives the normalized velocity as, for $\theta = \pi/2$, and $r/a \leqslant 1$,

$$\frac{q_{i\theta}}{u} = \left[\frac{3}{2} - 5\left(\frac{3}{\pi R_o}\right)^{1/2}\right]\left(2\frac{r^2}{a^2} - 1\right) + \left(\frac{2}{R_o}\right)^{1/2} f_i, \tag{3.1}$$

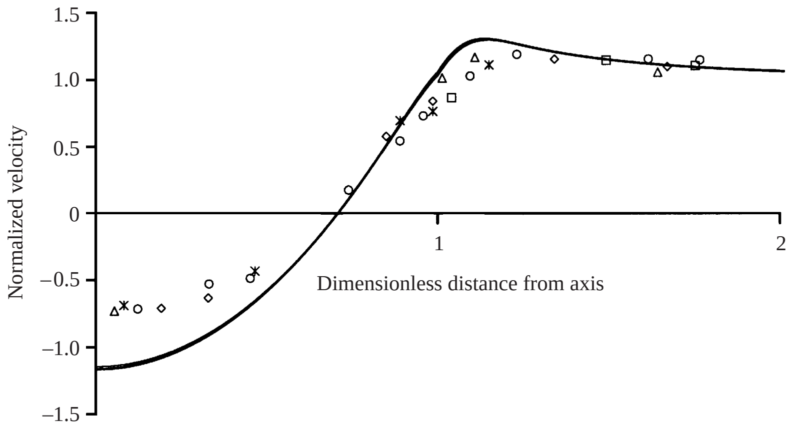

FIGURE 3. Vertical velocity measurements (symbols) of Bhaga & Weber (1981); together with the predictions of (3.1) and (3.5): thin line, $R_o = 188$; thick line, $R_o = 212$.

with the boundary layer correction given by

$$f_i = \frac{5}{4}\left(\frac{6}{\pi}\right)^{1/2} N\left[\frac{4}{9}, \left(\frac{R_o}{2}\right)^{1/2}\left(1-\frac{r}{a}\right)\right] - 5\Phi\left[\frac{3}{4}\left(\frac{R_o}{2}\right)^{1/2}\left(1-\frac{r}{a}\right)\right], \tag{3.2}$$

where $\Phi(z) = \pi^{-1/2}\exp(-z^2) - z\,\mathrm{erfc}(z)$ is the integral of the coerror function and

$$N(X, Y) = \frac{Y}{2\pi^{1/2}} \int_0^X \frac{\sin^2\theta(\lambda)}{(X-\lambda)^{3/2}} \exp\left[\frac{-Y^2}{4(X-\lambda)}\right] \mathrm{d}\lambda, \tag{3.3}$$

with $\sin^2\theta(\lambda)$ given by

$$\sin^2\theta(\lambda) = 1 - 4\cos^2\left[\frac{1}{3}\arccos\left(\frac{9}{4}\lambda - 1\right) + \frac{\pi}{3}\right]. \tag{3.4}$$

For $\theta = \pi/2$ and $r/a \geqslant 1$, Parlange (1970) predicts

$$\frac{q_{o\theta}}{u} = \left(1 + \frac{a^3}{2r^3}\right) + \left(\frac{2}{R_o}\right)^{1/2} f_o, \tag{3.5}$$

the first term being the standard potential flow and the second one the boundary layer correction with

$$f_o = -\frac{5}{4}\left(\frac{6}{\pi}\right)^{1/2} N\left[\frac{4}{9}, \left(\frac{R_o}{2}\right)^{1/2}\left(\frac{r}{a}-1\right)\right] - 5\Phi\left[\frac{3}{4}\left(\frac{R_o}{2}\right)^{1/2}\left(\frac{r}{a}-1\right)\right]. \tag{3.6}$$

The predictions of (3.1) and (3.5) are shown on figure 3. There is a marked discrepancy between the measurements and predictions near the axis ($r/a < 1/2$). Clearly, the results are quite insensitive to the value of R_o. The corresponding curves derived using (3.19) and (3.20) of Harper & Moore (1968) are shown in figure 4. Recall that those are far more complex than ours as another integral term has to be added and calculated numerically. Overall, there is good agreement between measurements and predictions, again with little variation due to changes in R_o. In contrast to figure 3, the results in figure 4 show better agreement with the data near the axis.

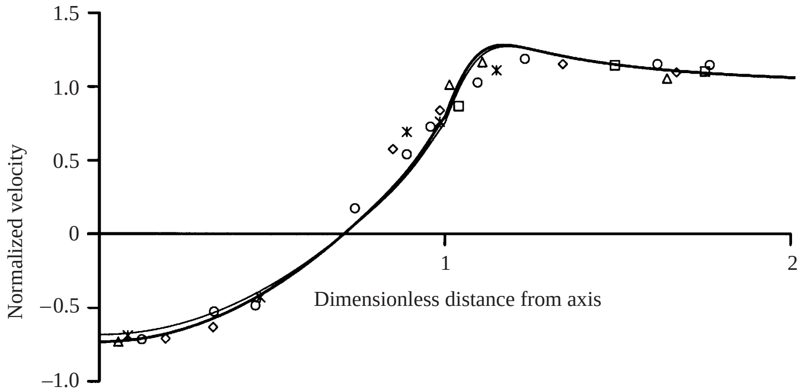

FIGURE 4. Vertical velocity measurements (symbols) of Bhaga & Weber (1981); together with the predictions of (3.19) and (3.20) from Harper & Moore (1968): thin line, $R_o = 188$; thick line, $R_o = 212$.

Close examination of figure 4 reveals a discontinuity in slope at $r/a = 1$. Likewise, this discontinuity is present in figure 3, although it is very difficult to discern because the correction terms in figure 3 are less than half those in figure 4. We shall consider this issue shortly.

So far we have considered the boundary layer correction at the periphery of the wake ($r/a \simeq 1$). When the boundary layers reach the rear of the sphere ($\theta = \pi$) they turn and form a thin wake behind the main wake for $r/a > 1$ and turn inwards for $r/a < 1$ to form a sort of inner wake. Those two thin wakes do not contribute significantly to the viscous dissipation since they affect a small volume only, the flow being axisymmetric. Thus, Parlange (1970) ignored them to calculate the drag. On the other hand, the velocity distribution is strongly affected in which case the inner wake cannot be ignored in its calculation.

As the inner wake progresses toward the front of the sphere it is slowly eroded by diffusion (figure 1). It is, we recall, the assumption of Parlange (1970) that in the front stagnation region the Hills's vortex is fully established whereas Harper & Moore (1968) assume that the wake is not affected by diffusion and that the signal is carried from the rear to the front essentially unchanged. There is no doubt that in the middle of the main wake ($\theta = \pi/2$) the inner wake is largely intact. Only in the front stagnation region could it be affected. It is then easy to estimate the velocity correction due to that inner wake at $\theta = \pi/2$. The calculation was carried out in detail by Moore (1963) for a spherical gas bubble, in which case he needed to consider only the thin wake outside the bubble and the f_o correction only contained the Φ term, see his equation (3.29). However, his procedure applies equally well to the inner wake and, at $\theta = \pi/2$, we obtain another correction to be added to the solution, equal to

$$-\frac{15}{R_o^{1/2}}\left\{\frac{1}{4}\left(\frac{3}{\pi}\right)^{1/2} N\left[\frac{8}{9},\left(\frac{R_o}{2}\right)^{1/2}\frac{r^2}{3}\right] - \Phi\left(R_o^{1/2}\frac{r^2}{8}\right)\right\}, \tag{3.7}$$

with a similar expression for the case of Harper & Moore (1968).

The result of this correction is shown in figure 5. The figure shows very good agreement between predictions and measured data, with little sensitivity to R_o. The corresponding correction for Harper & Moore (1968) is given in figure 6. The predictions there are still insensitive to R_o, with a clear slope discontinuity at $r/a = 1$.

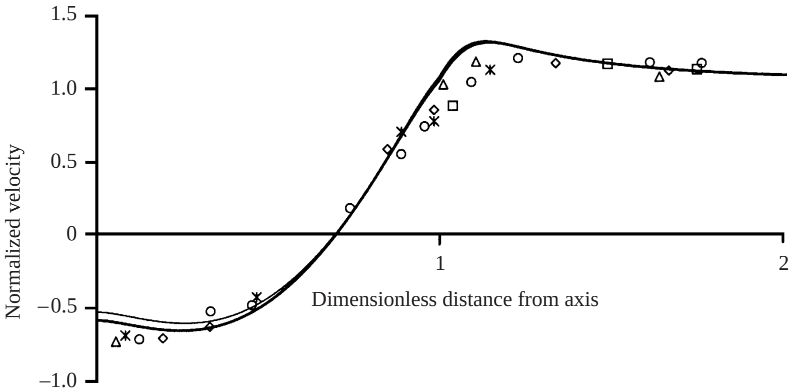

FIGURE 5. Vertical velocity measurements (symbols) of Bhaga & Weber (1981); together with the predictions of (3.1), (3.5) and (3.7): thin line, $R_o = 188$; thick line, $R_o = 212$.

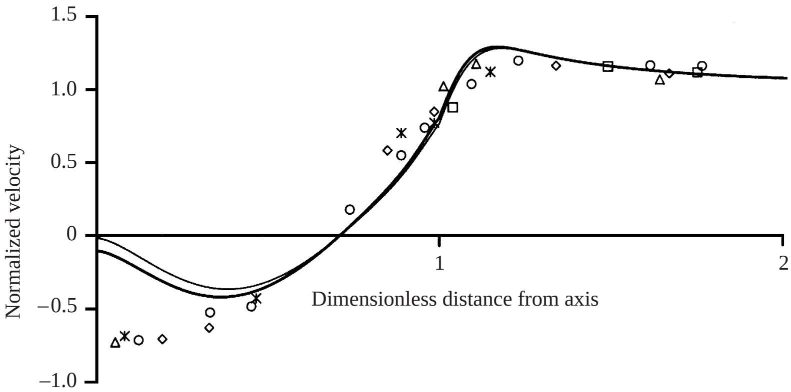

FIGURE 6. Vertical velocity measurements (symbols) of Bhaga & Weber (1981); together with the predictions of (3.19) and (3.20) from Harper & Moore (1968), and equivalent correction to that presented in (3.7): thin line, $R_o = 188$; thick line, $R_o = 212$.

The agreement between predictions and measurements is now quite poor near the axis ($r/a < 1/2$), compared with figure 4.

Several observations can be made in comparing the theoretical predictions in figures 5 and 6: Parlange (1970) shows an excellent fit to the data, providing perhaps a higher envelope for the observations. The prediction of Harper & Moore (1968) is less accurate especially for $1/2 < r/a < 1$. For $r/a < 1/2$ the inner wake correction greatly improves the result of Parlange (1970) when compared to figure 2, i.e. the Hill's vortex alone ignoring the inner wake. With the prediction of Harper & Moore (1968) the opposite happens: the inner wake destroys the apparent agreement in figure 2. In their case, the inner wake is crucial to their formulation and cannot be ignored. Finally at $r/a = 1$, we recall, as mentioned already, that for both predictions there is a discontinuity in slope and hence in stress, because the stress continuity is satisfied to the lowest order only (Harper & Moore 1968; Parlange 1970). This is particularly obvious for the prediction of Harper & Moore (1968) because their corrections are twice as large as those of Parlange (1970). Without affecting the solution to $O(R_o^{-1/2})$, we can easily remove this blemish if we multiply all the Φ corrections by an *ad hoc* factor: $1 - 8(3\pi R_o)^{-1/2}$ for Parlange (1970) and $1 + 4C(2/R_o)^{1/2}/5$ for (2.3) without significantly affecting the rest of the predictions. Indeed, plots of the slope-corrected predictions are very similar to those shown in figures 5 and 6, and so are not presented here.

4. Conclusions

The experiments of Bhaga & Weber (1981) provide crucial data to discriminate in general between the predictions of the theories of Harper & Moore (1968) and Parlange (1970) for recirculating flows within a fluid sphere, i.e. drops and bubbles. The agreement between the measured data and predictions of the latter theory is quite good, and there is certainly no experimental basis for the suggestion of Bhaga & Weber (1981) that the theory of Harper & Moore (1968) is in better agreement with observations. Their statement was based on an incomplete application of the theoretical results ignoring boundary layer corrections.

Finally, as the two competing theories are based on different behaviour of the vorticity along the axis of the fluid sphere – Harper & Moore (1968), in contrast to Parlange (1970), assuming that vorticity does not diffuse there – it is encouraging that they are in agreement in most places except near the axis. Even with a somewhat *ad hoc* correction to improve the theory of Harper & Moore (1968), the theory of Parlange (1970) remains clearly more accurate. In addition, computation of the latter's theory is far simpler because the flow field has no singularity. The corrections of $O(R_o^{-1/2})$ are less than half the magnitude of the corrections of Harper & Moore (1968). Since the theories apply for small corrections, this means that the theory of Parlange (1970) can be used for Reynolds numbers about five times smaller than that of Harper & Moore (1968).

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